Application of Hamilton's Law of Varying Action

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The Law of Varying Action enunciated by Hamilton in 1834-1835 permits the direct analytical solution of the problems of mechanics, both stationary and nonstationary, without consideration of force equilibrium and the theory of differential equations associated therewith. It has not been possible to obtain direct analytical solutions to nonstationary systems through the use of energy theory, which has been limited for 140 years to the principle of least action and to Hamilton's principle. It is shown here that Hamilton's law permits the direct analytical solution to nonstationary, initial value systems in the mechanics of solids without any knowledge or use of the theory of differential equations. Solutions are demonstrated for nonconservative, nonstationary particle motion, both linear and nonlinear.

Nomenclature

A_i = coefficients of power series
B_k = coefficients of power series
c = damping force coefficient; may be function of time
c_0 = damping coefficient at $t=t_0$
c_1 = rate of change of damping coefficient
F = force
g = gravitational parameter; taken as constant in this paper
k = spring force coefficient; may be function of time
k_0 = spring coefficient at $t=t_0$
k_1 = rate of change of spring coefficient
ℓ = length of pendulum arm
m = mass; may be function of time
m_0 = total mass at $t=t_0$
m_1 = rate of change of mass in variable mass problem; also mass number one in two-degree-of-freedom problem
M,N = number of terms in truncated power series
Q_i = general force acting in general displacement direction
$q_i, \dot{q}_i$ = dependent space variable, displacement, velocity
T = kinetic energy, work of inertial forces
$T_{1,2}$ = couples applied to pendulum bobs
t = real time
t_0 = time at which observation of phenomena begins
t_1 = time at which observation of phenomena ends
V_0 = initial velocity at $t=t_0$
W = work of all forces other than inertial forces
δ = operates on displacement while forces are held constant
τ = nondimensional time, $\tau=t/t_1$
ω = circular frequency, rad/sec, harmonic motion

Introduction

HAMILTON set forth the Law of Varying Action in papers concerning a general method in dynamics, published in 1834[1] and 1835[2]. When the varied paths were assumed to be coterminus with the real path in both space and time, Hamilton's law was reduced to Hamilton's principle, as pointed out in Ref. 3. By the year 1937, when Osgood published his text,[4] Hamilton's principle had been established as "...an independent foundation of mechanics."[4] Osgood simply postulated the integral

$$\int_{t_0}^{t_1} (T+W)\ \mathrm{d}t \tag{1}$$

which he called Hamilton's integral, and then obtained the following equation by varying Hamilton's integral and integrating the kinetic energy term by parts

$$\delta\int_{t_0}^{t_1} (T+W)\ \mathrm{d}t = \frac{\partial T}{\partial \dot{q}_i}\,\delta q_i\ \Big]_{t_0}^{t_1} - \int_{t_0}^{t_1}\left[\frac{\mathrm{d}}{\mathrm{d}t}\,\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} - Q_i\right]\delta q_i\ \mathrm{d}t \tag{2}$$

where generalized coordinates have been used instead of the Cartesian coordinates used by Osgood, and where δt has been set equal to zero at the outset. Following the concepts of the variational calculus, Osgood assumed all of the δq_i to vanish at t_0 and t_1; i.e., the end points of the varied paths are postulated to be coterminus with the real path; thus the first term on the right vanishes. The second term on the right is assumed to vanish because the integrand contains the equations of Lagrange, which are known to vanish. Equation (2) then reduces to Hamilton's principle

$$\delta\int_{t_0}^{t_1} (T+W)\ \mathrm{d}t = 0 \tag{3}$$

where it is understood that W is the work of both conservative and nonconservative forces.

When it is observed that the integrand of the right-hand integral of Eq. (2) vanishes whether or not the δq_i vanishes at t_0 and t_1, Eq. (2) results in the mathematical expression for the Law of Varying Action

$$\delta\int_{t_0}^{t_1} (T+W)\ \mathrm{d}t - \frac{\partial T}{\partial \dot{q}_i}\ \Big]_{t_0}^{t_1} = 0 \tag{4}$$

The term $(\partial T/\partial \dot{q}_i)\delta q_i$ does not in general vanish at t_1, as taught by the variational calculus. Note that the zero on the right-hand side of Eq. (4) results from the fact that the integral on the right-hand side of Eq. (2) vanishes. It has nothing to do with the proof in the variational calculus that the integral on the left-hand side is an extremum.

It has been obvious to competent researchers that something is wrong somewhere. Bisplinghoff and Ashley (Ref. 5, p. 36) mention the problem: "No difficulty is encountered when Lagrange's equations can be constructed, for these are differential equations which may, in principle, be integrated from instant to instant. But the question of how to

Received March 19, 1973; revision received March 20, 1975. Supported in part by the NASA Langley Research Center, Grant NGR 36-008-197, and in part by The Ohio State University, Department of Aeronautical and Astronautical Engineering. Calculations were made by D. P. Beres and E. L. Grau, Graduate Associates.
Index category: Structural Dynamic Analysis.
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handle the upper limit t_1 during direct application of Hamilton's principle is a more subtle one." The question, however, is not in "how to handle the upper limit t_1."

Fung Ref. 6 p. 318, arrives at Hamilton's Law for a deformable body, although, as is the common practice, he terms it Hamilton's principle: "In some applications of the direct method of calculation it is even desirable to liberalize the variations δu_i at the instant t_0 and t_1 and use Hamilton's principle in the variational form, Eq. (4), which cannot be expressed elegantly as the minimum of a well-defined functional. On the other hand, such a formulation will be accessible to the direct methods of solution. On introducing Eqs. (5), (7), and (10), we may rewrite Eq. (4) in the following form:

$$(13)\quad \int_{t_0}^{t_1} \delta(U-K+A)\,\mathrm{d}t = \int_{t_0}^{t_1}\int_V F_i \delta u_i \mathrm{d}V\mathrm{d}t + \int_{t_0}^{t_1}\int_S T_i\delta u_i \mathrm{d}S\mathrm{d}t + \int_V \rho\,\frac{\partial u_i}{\partial t}\,\delta u_i \mathrm{d}V]_{t_0}^{t_1}\,."$$

First, Fung's Eq. (13) is not Hamilton's principle because of its last term. Second, Fung gives no indication as to how the problem of the upper limit t_1 as pointed out by Bisplinghoff and Ashley, is to be treated. Third, it must be assumed that the last term of Eq. (13) would be treated by Fung in accordance with the concepts of the variational calculus, which means that it ultimately would be set equal to zero. And fourth, this writer is not aware of any successful attempt to achieve a direct solution to any nonstationary problem of mechanics by the use of the concepts of the variational calculus applied to Fung's Eq. (13). For stationary problems, his Eq. (13) reduces immediately, as does Hamilton's law, to Hamilton's principle for which

$$(\partial T/\partial \dot{q}_i)\ \delta q_i]_{t_0}^{t_1}=0$$

It is the purpose of this paper to demonstrate the application of Hamilton's law to achieve completely general analytical solutions to nonstationary particle motion by direct application of Eq. (4). The foundation on which this work rests is time-space continuity.[7] Two well-known observations are made: 1) The path of any mass particle through time-space is continuous; i.e., no particle of matter can occupy two points in space at the same instant in time. 2) The slope of the time-space path of any particle is also continuous.

Thus, the time-space path of any particle of matter is continuous, with continuous first derivatives. This is precisely part of the requirement for admissible functions when using the theory of Ritz[8] in conjunction with stationary problems in the theory of elasticity. As will be seen, however, when a system is defined, time intervals t_0, t_1 may be chosen such that the time-space path and all of its derivatives are continuous within the chosen interval. Thus, the functional relationship between the dependent space variables and the independent time variable may be determined from a power series, in principle, exactly. In practice, when the function is an algebraic function, it may be, in fact, determined exactly. When the function is a transcendental function, it may be determined "exactly" in the same sense that the "exact" answer to any transcendental function may be determined by a finite number of operations. No theory of differential equations is required.

Discussion

Consider a particle of constant mass on which acts a linear restoring force $k(t)q$, a linear viscous damping force $c(t)\dot{q}$ and a force that is a specified function of time, $F(t)$. In other words, this is the linear forced-damped-spring-mass system for which the differential equation is well known and to which general solutions for comparison of results may be readily obtained from the theory of differential equations. By substituting the kinetic energy and the work of the prescribed forces into Eq. (4) and taking the variation (as in virtual work, which predated the variational calculus by many years), one obtains the equation

$$\int_{t_0}^{t_1} [m\dot{q}\,\delta\dot{q} + (F-kq-c\dot{q})\delta q]\mathrm{d}t - m\dot{q}\,\delta q]_{t_0}^{t_1}=0 \tag{5}$$

The conventional procedure at this point is to derive the differential equation from which the solution may be obtained. The purpose of this paper, however, is to demonstrate that the solution may be obtained without any reference to or knowledge of differential equations. By virtue of the fact that, within the interval t_0, t_1, the time-space path is continuous with continuous derivatives a simple truncated power series is an admissible function

$$q=q_0+V_0t+\sum_{i=2}^{N}\bar{A}_it^i \tag{6}$$

Note that no concept of the shape of the time-space path is necessary. The power series must satisfy the specified displacement and velocity at $t=t_0$ but not at $t=t_1$. The displacement and velocity at t_1 cannot in general be known in advance because they are the result of both the initial conditions and the time history of the forces acting between t_o and t_1.

In Eq. (5), m, k, and c may be any one or all functions of time. However, for the present, assume these parameters to be constant. Let, $t=t_1\tau$ and divide by m/t_1^2. Let the instant in time t_0 at which the observation begins be $t_0=0$. Equation (5) becomes

$$\int_0^1 [\dot{q}\delta\dot{q} - \frac{ct_1}{m}\,\dot{q}\delta q - \frac{Kt_1^2}{m}\,q\delta q + \frac{t_1^2}{m}\,F(t,\tau)\,\delta q]\,\mathrm{d}\tau - \dot{q}\,\delta q]_0^1=0 \tag{7}$$

In terms of the nondimensional time τ the admissible function is simply

$$q=q_0+V_0t_1\tau+\sum_{i=2}^{N}A_i\tau^i \tag{8}$$

Substitute Eq. (8) into Eq. (7). Integrate to obtain a set of algebraic equations. These equations expressed in matrix form are

$$\left[\frac{ij}{i+j-1} - i - \frac{ct_1}{m}\,\frac{i}{i+j} - \frac{kt_1^2}{m}\,\frac{1}{i+j+1}\right]\left\{A_i\right\} = \left[-\frac{t_1^2}{m}\int_0^1 F(t_1\tau)\,\tau^j\mathrm{d}\tau + \frac{(cV_o+kq_o)}{m(j+1)}\,t_1^2 + \frac{kt_1^3V_o}{m(j+2)}\right] i,\ j=2,3,\ldots,N \tag{9}$$

Equation (9) constitutes the general analytical solution to the system over the interval, $0\leq\tau\leq 1$, in that the time-space path yielded by the solution to this matrix equation is the sum of the particular and complementary solutions that would be obtained from the differential equation. Note the initial conditions as well as the damping coefficient appear in the nonhomogeneous term. When these parameters, along with $F(t_1\tau)$ and t_1, are specified, the coefficients of the power series may be obtained. In particular, note that even with $F=0$, the equations are not homogeneous, as in the case of the homogeneous differential equation.

Both nonstationary problems and simple harmonic motion can be treated with equal ease. It may be of interest to point out that if $F(t_1\tau)$, c, and V_0 are set equal to zero, Eq. (9) will generate a cosine function. If $F(t_1\tau)$, c, and q_0 are set equal to zero, Eq. (9) will generate a sine function. These, of course, are precisely the functions defined by the differential equation of the simple spring-mass system. The important point to be made is this: without any knowledge of the mathematical functions involved in the answer, Eq. (4) generates from the power series whatever function is required to yield the solution.

Forcing Function Applied to a Linear System with Variable Mass, Damping, and Spring Force

Numerical integration is used to evaluate the matrix elements in the event that the integrands are defined by, say, curves generated from test data. However, to illustrate the generality without getting into such details, the following functions are assumed:

$$m(t)=m_0-m_1 t \quad c(t)=c_0+c_1 t$$

$$k(t)=k_0+k_1 t \quad F=F_0-m_1\dot{q}-(m_0-m_1 t)q$$

The matrix equation obtained from Eq. (5) is,

$$\left[m_0\frac{i(1-i)}{i+j-1}+t_1\frac{m_1 i(1-i)-c_0 i}{i+j}-t_1^2\frac{c_1 i+k_0}{i+j+1}\right.$$

$$\left.-t_1^3\frac{k_1}{i+j+2}\right]\left\{A_i\right\}=\left[t_1^2\frac{m_0 g-F_0+c_0V_0+k_0q_0}{j+1}\right.$$

$$+t_1^3\frac{c_1V_0+k_0V_0+k_1q_0-mg}{j+2}$$

$$\left.+t_1^4\frac{k_1V_o}{j+3}\right]i,j=2,3,\ldots,N \tag{10}$$

Figure 1 shows the resulting displacement and velocity for one choice of parameters. The differential equation is also shown in Fig. 1. Substitution of the direct analytical solution for q, $\dot{q}$, and $\ddot{q}$ into the differential equation shows equilibrium of the forces to be satisfied within less than 4/10,000 of 1%.

Nonlinear, Nonconservative, Nonstationary Double Pendulum

Every student who has taken a course in intermediate mechanics, in which Lagrange's equation was introduced, has derived the nonlinear differential equations of motion for this classical system. The analytical solution, without any reference to these nonlinear differential equations, is demonstrated. Figure 2 shows the coordinates selected. For this coordinate system and the specified generalized forces acting (a damping couple at each hinge, an elastic restoring couple at each hinge, a step couple at the support hinge, and gravity), the kinetic energy and work are given by

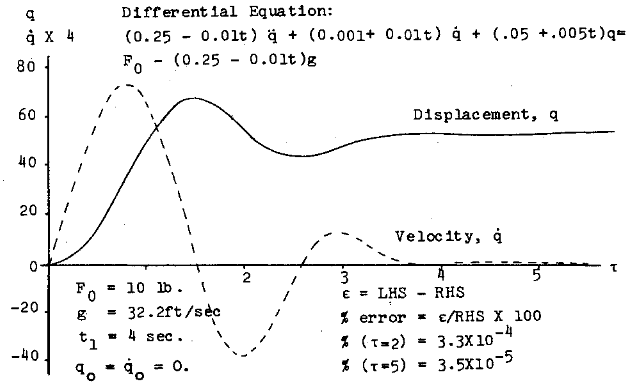

Fig. 1 Linear, nonstationary system.

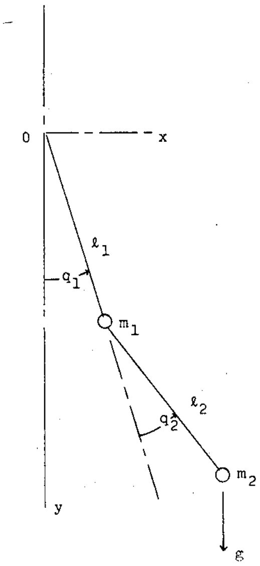

Fig. 2 Coordinates, nonlinear system.

$$T=\tfrac{1}{2}\left[(m_1+m_2)\ell_1^2+m_2\ell_2^2)\right]\dot{q}_1^2+\tfrac{1}{2}m_2\ell_2^2\dot{q}_2^2$$

$$+m_2\ell_1\ell_2\dot{q}_1^2\cos q_2+m_2\ell_2\dot{q}_1\dot{q}_2\,(\ell_2+\ell_1\cos q_2) \tag{11}$$

$$W=\left[-c_1\dot{q}_1-k_1q_1+T_1(t)\right]q_1+m_1g\ell_1\cos q_1$$

$$+\left[-c_2\dot{q}_2-k_2q_2+T_2(t)\right]q_2+m_2g\ell_1\cos q_1$$

$$+m_2g\ell_2\cos(q_1+q_2) \tag{12}$$

Substitute Eqs. (11) and (12) into Eq. (4). Carry out the indicated operations with

$$q_1=q_{01}+\dot{q}_{01}t_1\tau+\sum_{i=2}^{N}A_1\tau^i \tag{13}$$

and

$$q_2=q_{02}+\dot{q}_{02}t_1\tau+\sum_{k=2}^{M}B_k\tau^k \tag{14}$$

This yields a set of nonhomogeneous algebraic equations, which may be written in matrix form:

$$\begin{bmatrix} M_{ij} & M_{kj} \\ M_{is} & M_{ks}\end{bmatrix}\begin{Bmatrix} A_i \\ B_k\end{Bmatrix}=\begin{Bmatrix} F_j \\ F_s\end{Bmatrix} \tag{15}$$

The matix elements are given in the appendix. Equation (15) consitutes the general solution to the defined problem in the same sense that the general solution to a linear differential equation is defined as the sum of the "complementary" solution and the "particular" solution. The principle of superposition does not apply to the nonlinear system, and is not needed nor used for linear systems with the theory demonstrated here. To obtain the solution (i.e., the time-space path of each of the particles), specify the various parameters, along with the time interval t_1 over which the truncated power series are to give the functional forms of the time-space paths. Equation (15) is then solved by direct iteration, or by any suitable method. Let

$$t_1=\beta 2\pi\ell_1/g=1.0 \qquad m_2\ell_2^2/m_1\ell_1^2=1.0$$

$$c_1t_1/m_1\ell_1^2=1.6\pi\beta \qquad c_2t_1/m_1\ell_1^2=1.6\pi\beta$$

$$k_1t_1^2/m_1\ell_1^2=1.2861328\pi^2\beta^2 \qquad k_2t_1^2/m_1\ell_1^2=22.5701828\pi^2\beta^2$$

$$T_1t_1^2/m_1\ell_1^2=2\pi^2\beta^2 \qquad T_2t_1^2/m_1\ell_1^2=0$$

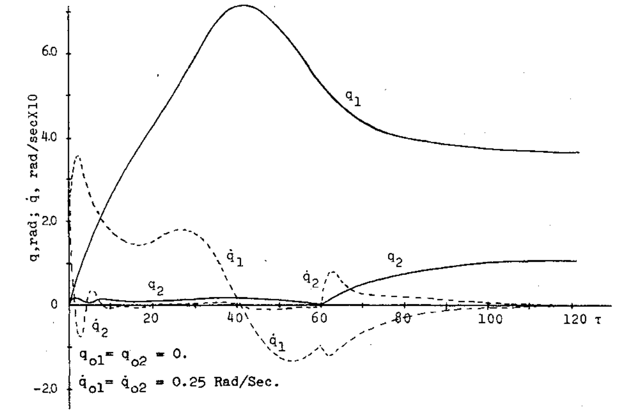

Fig. 3 Nonlinear, nonstationary system.

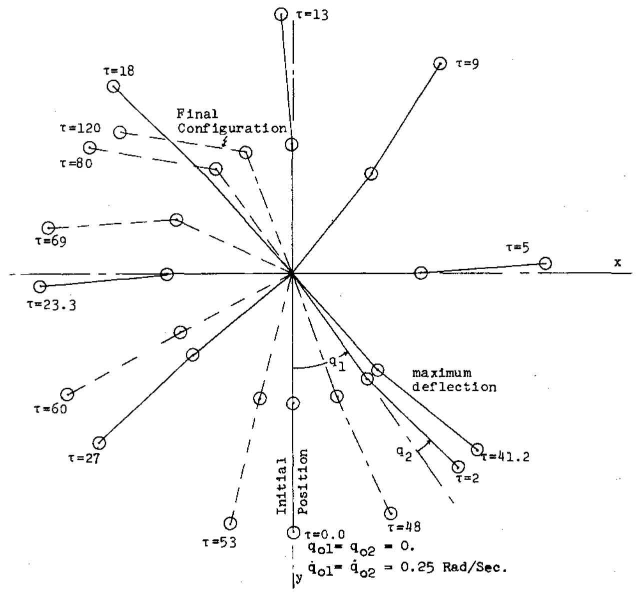

Fig. 4 Configurations: nonlinear, nonstationary system.

β is a number chosen to give the desired amount of real time in the interval t_0, t_1. The peculiar choice of the spring and damping constants resulted from a desire by the author to have the motion end in a prescribed configuration. With these quantities specified, the matrix elements in Eq. (15) may readily be calculated from the Appendix. The coefficients of the power series are then obtained from Eq. (15). When it was noted that the extremely "stiff" spring on the second pendulum was causing the relative motion of the second pendulum to be small, the spring constants were changed to $2k_1$ and $k_2/5$ and the calculation continued. The results (time-space paths and velocities of the two particles) are shown in Fig. 3. The configuration, at various instants in time, is shown in Fig. 4. Note that this analytical solution, purely from the energy equation, has described the motion completely from its onset until the system has come to a state of rest (including discontinuities in $\ddot{q}_1$ and $\ddot{q}_2$ when the spring constants were changed). This state of rest can be verified by the equation of static equilibrium. Since the energy is not constant during the motion, the solution can be checked at any instant of time by direct substitution into the differential equations. This requires no knowledge of the theory of such equations.

Conclusion

It has been shown that, contrary to the state of energy theory found in textbooks and in the variational calculus, direct analytical solutions to nonstationary particle motion may be obtained through application of Hamilton's Law for nonstationary, nonconservative systems. No knowledge is required as to whether a system is conservative, nonconservative, stationary, or nonstationary. As demonstrated, Hamilton's Law of Varying Action permits direct generation of the functions representing the motion of complicated systems with simplicity, generality, and accuracy. Further, as with any complete and general analytical solution, the functions thus generated yield all information available from the laws of mathematics, without reference to the theory of differential equations.

Appendix

Matrix elements for the linear or nonlinear, conservative, or nonconservative double pendulum

$$M_{ij} = -[1+(m_1/m_2)+m_2\ell_2^2/m_1\ell_1^2]\,[i(i-1)/(i+j-1)] - (c_1t_1/m_1\ell_1^2)\,[(i/(i+j)] - (k_1t_1^2/m_1\ell_1^2)\,[1/(i+j+1)]$$

$$M_{kj} = -(m_2\ell_2^2/m_1\ell_1^2)\,[k(k-1)/(k+j-1)]$$

$$M_{is} = -(m_2\ell_2^2/m_1\ell_1^2)\,[i(i-1)/(i+s-1)]$$

$$M_{ks} = -(m_2\ell_2^2/m_1\ell_1^2)\,[k(k-1)/(k+s-1)] - (c_2t_1/m_1\ell_1^2)\,[k/(k+s)] - (k_2t_1^2/m_1\ell_1^2)\,[1/(k+s+1)]$$

$$\begin{aligned} F_j = & -(t_1^2/m_1\ell_1^2)\int_0^1 T_1(\tau)\,\tau^j d\tau \\ & +[1+(m_2/m_1)]\,(gt_1^2/\ell_1) \\ & \times \int_0^1 \tau^j \sin q_1 d\tau + (m_2\ell_2 g t_1^2/m_1\ell_1^2) \\ & \times \int_0^1 \tau^j \sin(q_1+q_2)d\tau \\ & +(m_2\ell_2/m_1\ell_1)\int_0^1 \tau^j \dot{q}_2^2 \sin q_2 d\tau \\ & +(m_2\ell_2/m_1\ell_1)\int_0^1 \tau^j(2\ddot{q}_1+\ddot{q}_2)\cos q_2 d\tau \\ & +2(m_2\ell_2/m_1\ell_1)\int_0^1 \tau^j \dot{q}_1\dot{q}_2 \sin q_2 d\tau \end{aligned}$$

$$\begin{aligned} F_s = & -(t_1^2/m_1\ell_1^2)\int_0^1 \tau^s T_2(\tau)d\tau \\ & +(m_2\ell_2 g t_1^2/m_1\ell_1^2)\int_0^1 \tau^s \sin(q_1+q_2)\,d\tau \\ & +(m_2\ell_2/m_1\ell_1)\int_0^1 \tau^s \ddot{q}_1 \cos q_2 d\tau \\ & +(m_2\ell_2/m_1\ell_1)\int_0^1 \tau^s \dot{q}_1^2 \sin q_2 d\tau \end{aligned}$$

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